

# QUANTUM PARTICLE BEHAVIOR IN CLASSICALLY SINGULAR SPACETIMES

D. A. KONKOWSKI

*Department of Mathematics, U.S. Naval Academy  
Annapolis, Maryland, 21012, USA  
E-mail: dak@usna.edu*

T.M. HELLIWELL

*Department of Physics, Harvey Mudd College  
Claremont, California, 91711, USA  
E-mail: helliwell@HMC.edu*

We review the classical and quantum singularity structure of a broad class of spacetimes with asymptotically power-law behavior near the origin. Quantum considerations “heal” a large class of scalar curvature singularities.

## 1. Introduction

The question addressed in this review is: What happens if instead of classical particle paths (time-like and null geodesics) one uses quantum mechanical particles to identify singularities? The answer for any asymptotically power-law space-time is given. This conference proceeding is based on articles by the authors<sup>1</sup> and by K. Lake.<sup>2</sup>

## 2. Types of Singularities

### 2.1. Classical Singularities

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration<sup>3,4</sup> in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are three different types of singularity:<sup>5</sup> quasi-regular, non-scalar curvature and scalar curvature. Whereas quasi-regular singularities are topological, curvature singularities are indicated by diverging components of the Riemann tensor when it is evaluated in a parallel-propagated orthonormal frame carried along a causal curve ending at the singularity.

### 2.2. Quantum Singularities

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity.<sup>7</sup> Technically, a static ST (spacetime) is QM-singular if the spatial portion of the Klein-Gordon operator is not essentially self-adjoint on  $C_0^\infty(\Sigma)$  in  $L^2(\Sigma)$  where  $\Sigma$  is a spatial slice.

### 3. Asymptotically Power-Law Spacetimes

We consider a class of spacetimes that can be written in power-law metric form in the limit of small  $r$ ,

$$ds^2 = -r^\alpha dt^2 + r^\beta dr^2 + C^{-2} r^\gamma d\theta^2 + r^\delta (dz + Ad\theta)^2 \quad (1)$$

where  $\beta, \gamma, \delta, C, A$  are constant parameters and the variables have the usual ranges. We are particularly interested in the metrics at small  $r$ , because we suppose that if the spacetime has a classical curvature singularity (and nearly all of these do), it occurs at  $r = 0$ .<sup>a</sup>

We can eliminate  $\alpha$  by rescaling  $r$  which results in two separate metric types:

- Type I:

$$ds^2 = r^\beta (-dt^2 + dr^2) + C^{-2} r^\gamma d\theta^2 + r^\delta (dz + Ad\theta)^2 \quad \alpha \neq \beta + 2. \quad (2)$$

- Type II:

$$ds^2 = -r^{\beta+2} dt^2 + r^\beta dr^2 + C^{-2} r^\gamma d\theta^2 + r^\delta (dz + Ad\theta)^2 \quad \alpha = \beta + 2. \quad (3)$$

### 4. Classical Singularity Analysis

Except for isolated values of  $\beta, \gamma, \delta, C, A$  all of these power-law spacetimes have diverging scalar polynomial invariants if and only if  $\beta > -2$ .

#### 4.1. Type I Spacetimes

Lake<sup>2</sup> has shown that in Type I STs  $r = 0$  is timelike, naked and at a finite affine distance if and only if  $\beta > -1$ , implying that there is a classical singularity at  $r = 0$  if and only if  $\beta > -1$ .

#### 4.2. Type II Spacetimes

Likewise, Lake<sup>2</sup> has shown that in Type II STs  $r = 0$  is null, naked and at a finite affine distance and thus is a classical singularity for all  $\beta > -2$ .

### 5. Quantum Singularity Analysis

To study the quantum particle propagation in these spacetimes (for simplicity, we take  $A = 0$ ), we use massive scalar particles described by the Klein-Gordon equation and the "limit point - limit circle" criterion of Weyl.<sup>8,9</sup> This means that, in

---

<sup>a</sup>If  $\alpha = \beta = \gamma = \delta = 0$ ,  $C \neq 1$  indicates a quasi-regular singularity (a disclination) and  $A \neq 0$  indicates a quasi-regular singularity (a dislocation) (see, e.g., Konkowski and Helliwell<sup>6</sup>).

particular, we study the radial equation in a one-dimensional Schrödinger form with a 'potential' and determine the number of solutions that are square integrable. If we obtain a unique solution, without placing boundary conditions at the location of the classical singularity, we can then say that the Klein-Gordon operator is essentially self-adjoint and the spacetime is QM-nonsingular.

### 5.1. Type I Spacetimes

There is a quantum singularity "bowl" in parameter space for these metrics. The bowl is bounded by (1) a bottom which is formed from a  $\beta = -2$  base plane and (2) the sides which are composed of (a) two vertical planes with  $\gamma + \delta = 6$  and  $\gamma + \delta = -2$  and (b) two tilted planes with  $\delta = \beta + 2$  and  $\gamma = \beta + 2$ . Points within the bowl are QM singular; points outside the bowl are QM non-singular.

### 5.2. Type II Spacetimes

Type II STs are globally hyperbolic; the wave operator in this case must be essentially self-adjoint, so these spacetimes contain no quantum singularities. It is easy to verify this conclusion directly by checking the essential self-adjointness of the wave operator using the "limit point - limit circle" technique.

## 6. Conclusions

A large class of classically singular asymptotically power-law spacetimes has been shown to be quantum mechanically non-singular. Invoking an energy condition (e.g., weak or strong) can eliminate more singular spacetimes, but no choice completely eradicates them.

## 7. Acknowledgments

One of us (DAK) thanks Queen Mary, University of London, where some of this work was carried out.

## References

1. T.M. Helliwell and D.A. Konkowski, *Class. Quantum Grav.* **24**, 3377 (2007).
2. K. Lake, *Gen. Rel. Grav.* **40** 1609 (2008).
3. S.W. Hawking and G.F.R. Ellis, *The Large-Scale Structure of Spacetime* (Cambridge University Press, 1973).
4. R. Geroch, *Ann. Phys.* **48**, 526 (1968)
5. G.F.R. Ellis and B.G. Schmidt, *Gen. Rel. Grav.* **8**, 915 (1977).
6. D.A. Konkowski and T.M. Helliwell, *Gen. Rel. Grav.* **38**, 1069 (2006).
7. G.T. Horowitz and D. Marolf, *Phys. Rev. D* **52**, 5670 (1995).
8. M. Reed and B. Simon, *Functional Analysis* (Academic Press, 1972); M. Reed and B. Simon, *Fourier Analysis and Self-Adjointness* (Academic Press, 1972).
9. H. Weyl, *Math. Ann.* **68**, 220 (1910).